

# Asymptotic Cosmological Behavior of Scalar-Torsion Mode in Poincaré Gauge Theory

Chao-Qiang Geng<sup>1,2,3a</sup>, Chung-Chi Lee<sup>2b</sup> and Huan-Hsin Tseng<sup>2c</sup>

<sup>1</sup>*College of Mathematics & Physics,*

*Chongqing University of Posts & Telecommunications, Chongqing, 400065, China*

<sup>2</sup>*Department of Physics, National Tsing Hua University, Hsinchu, Taiwan 300*

<sup>3</sup>*Physics Division, National Center for Theoretical Sciences, Hsinchu, Taiwan 300*

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## Abstract

We study the cosmological effect of the simple scalar-torsion ( $0^+$ ) mode in Poincaré gauge theory of gravity. We find that for the non-constant (affine) curvature case, the early evolution of the torsion density  $\rho_T$  has a radiation-like asymptotic behavior of  $a^{-4}$  with  $a$  representing the scale factor, along with the stable point of the torsion pressure ( $P_T$ ) and density ratio  $P_T/\rho_T \rightarrow 1/3$  in the high redshift regime ( $z \gg 0$ ), which is different from the previous result in the literature. We use the Laurent expansion to resolve the solution. We also illustrate our result by the execution of numerical computations.

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<sup>a</sup> geng@phys.nthu.edu.tw

<sup>b</sup> g9522545@oz.nthu.edu.tw

<sup>c</sup> d943335@oz.nthu.edu.tw

## I. INTRODUCTION

The recent cosmological observations, such as those from type Ia supernovae [1, 2], cosmic microwave background radiation [3, 4], large scale structure [5, 6] and weak lensing [7], reveal that our universe is subject to a period of acceleration. In general, there are two ways to explain the phenomenon of the late-time accelerating universe [8] either by modifying the left- and right-handed sides of Einstein equation, called modified gravity and modified matter theories, respectively. For modified gravity theories, the acceleration is accounted as a part of the gravitational effect, while modified matter theories are constructed by including some negative pressure matter that could result in the expanding effect. In this study, we adopt the view point of an alternative gravity theory by selecting so-called Poincaré gauge theory (PGT) [9–11], which is also suitable to describe the late-time accelerating behavior [12, 13].

PGT is under the consideration of gauging Poincaré group  $P_4 = \mathbb{R}^{1,3} \rtimes O(1, 3)$  for gravity, and it sets out from a Riemann-Cartan spacetime  $(M, g, \nabla)$ , where  $M$  is a differentiable manifold,  $g$  is a metric on  $M$ , and  $\nabla$  is a general affine metric-compatible connection with  $\nabla g \equiv 0$  so that it has a canonical decomposition into  $\nabla = \overline{\nabla} + K$  with  $\overline{\nabla}$  the Riemannian part and  $K$  the contortion tensor written in terms of the torsion tensor  $T$  of  $\nabla$ . As a result, PGT is in general a theory of gravitation [9, 10] with torsion, which couples to the spin source. The theory comprises a degenerate case of the Einstein's general relativity (GR) of the vanishing torsion  $T$  and the Einstein-Cartan theory [14] with a torsion field equation (TFE) algebraically coupled to the intrinsic spin of the source, resulting in a non-dynamical torsion field.

In this work, we concentrate on a specific quadratic theory with only the scalar-torsion mode in PGT, which possesses the dynamical torsion field. This particular mode is the *simple*  $0^+$  *mode*, which is one of the six modes:  $0^\pm, 1^\pm$  and  $2^\pm$  labeled by spin and parity, based on the linearized theory [15, 16]. We remark that the  $0^-$  mode interacts with intrinsic spins of fermions [12, 17]. However, its contribution is considered to have largely diminished from the early universe to the present time so that its effect in the current stage must be slight. On the other hand, since the  $0^+$  mode has no interaction with any fundamental source [12, 18], one could imagine that it remains to have a considerable portion within the current universe. Consequently, this mode naturally becomes the subject to study [19].

In view of the FLRW cosmology, the vanishing spin current of the scalar-torsion mode

renders a set of nonlinear equations which address the evolutions of the metric and torsion field. Under the positivity energy argument, Shie, Nester and Yo (SNY) in [12] have observed two separate cases: one has a constant affine curvature ( $R \equiv \text{const}$ ) which violates the positive energy condition; and the other subject to the condition forms a system of nonlinear ordinary differential equations (ODEs). In the former, it has a late time de Sitter space asymptote [20] yet the torsion energy density could be negative, and the equation of state (EoS) of the torsion field has an interesting behavior [13]. In the latter, no obvious analytic solution is found so that numerical methods are generally applied. In particular, one of interesting features in our previous study [13] for the latter case is that the torsion EoS has an asymptotic behavior in the high redshift regime, whereas some other studies in the literature [12, 20] point out that the affine curvature  $R$ , torsion scalar  $\Phi$  and Hubble parameter  $H = \dot{a}/a$  are oscillatory during the cosmological evolution. Since the oscillating behaviors do not appear in our work [13], a thorough study is clearly needed. In this paper, we find a proof from a semi-analytical solution in the large curvature regime to support our non-oscillatory result. In order to demonstrate the conformity with our semi-analytical solution, we will also present the numerical analysis.

## II. SCALAR-TORSION COSMOLOGY

### A. Formulation

In this note, we explore a specific scalar-torsion mode in PGT called the *simple*  $0^+$  *mode*, given by [12]

$$L_{\text{SNY}} = \frac{a_0}{2}R + \frac{b}{24}R^2 + \frac{a_1}{8}(T_{ijk}T^{ijk} + 2T_{ijk}T^{kji} - 4T_kT^k), \quad (1)$$

with the positive coefficients of  $a_{0,1}$  and  $b$  are required by the positivity energy argument. Under the FLRW metric, the field equations have been shown in Eqs. (2.11)-(2.13) in Ref. [13]. The Friedmann equations for the scalar-torsion mode are given by

$$\begin{aligned} H^2 &\equiv \frac{\rho_c}{3a_0} \equiv \frac{\rho_M + \rho_T}{3a_0}, \\ \dot{H} &\equiv -\frac{\rho_c + p_{\text{tot}}}{2a_0} \equiv -\frac{\rho_c + p_M + p_T}{2a_0}, \end{aligned} \quad (2)$$

where  $a_0 = (8\pi G)^{-1}$  in GR, the subscript  $M$  represents the ordinary matter including both dust ( $m$ ) and radiation ( $r$ ), and  $\rho_T$  and  $p_T$  correspond to the torsion density and pressure

of the effective geometric effect other than GR, defined as

$$\begin{aligned}\rho_T &= 3\mu H^2 - \frac{b}{18} \left( R + \frac{6\mu}{b} \right) (3H - \Phi)^2 + \frac{b}{24} R^2, \\ p_T &= \frac{\mu}{3} (R - \bar{R}) + \frac{\rho_T}{3},\end{aligned}\tag{3}$$

respectively. From (3), we can discuss the torsion EoS,  $w_T$ , defined by [13]

$$w_T = \frac{p_T}{\rho_T}.\tag{4}$$

## B. Semi-analytical solution in high redshift

In this context, we provide the semi-analytical solution of the positive energy scalar-torsion mode in the large scalar affine curvature limit  $R \gg 6\mu/b$  which is commonly achieved in the high redshift regime ( $a \ll 1$ ). In such situation, we write the energy density of ordinary matter and torsion in series expansion of  $a(t)$  as

$$\begin{aligned}\rho_M &= \frac{\rho_m^{(0)}}{a^3} + \frac{\rho_r^{(0)}}{a^4}, \\ \frac{\rho_T}{\rho_m^{(0)}} &= \sum_{k=-c}^{\infty} A_{-k} a^k,\end{aligned}\tag{5}$$

respectively.

Before the analysis, we first follow the rescaling of the parameters in [13],

$$\begin{aligned}\tilde{a}_0 &= a_0/m^2 b, \quad \tilde{a}_1 = a_1/m^2 b, \quad \tilde{t} = t \cdot m, \quad \tilde{\mu} = \tilde{a}_0 + \tilde{a}_1, \\ \tilde{H}^2 &= H^2/m^2, \quad \tilde{\Phi} = \Phi/m, \quad \tilde{R} = R/m^2,\end{aligned}\tag{6}$$

where  $m^2 = \rho_m^{(0)}/3a_0$ . The equations of motion [13] can be rewritten as dimensionless equations:

$$\frac{d\tilde{H}}{d\tilde{t}} = \frac{\tilde{\mu}}{6\tilde{a}_1} \tilde{R} - \frac{\tilde{a}_0}{2\tilde{a}_1 a^3} - 2\tilde{H}^2,\tag{7}$$

$$\frac{d\tilde{\Phi}}{d\tilde{t}} = \frac{\tilde{a}_0}{2\tilde{a}_1} \left( \tilde{R} - \frac{3}{a^3} \right) - 3\tilde{H}\tilde{\Phi} + \frac{1}{3}\tilde{\Phi}^2,\tag{8}$$

$$\frac{d\tilde{R}}{d\tilde{t}} \simeq -\frac{2}{3}\tilde{R}\tilde{\Phi},\tag{9}$$

$$\frac{\tilde{R}}{18} \left( 3\tilde{H} - \tilde{\Phi} \right) - \frac{\tilde{R}^2}{24} - 3\tilde{a}_1 \tilde{H}^2 = 3\tilde{a}_0 \left( \frac{1}{a^3} + \frac{\chi}{a^4} \right),\tag{10}$$

respectively, where  $\chi = \rho_r^{(0)}/\rho_m^{(0)}$ . In (9), we have taken the approximation of  $R \gg 6\mu/b$  for the high redshift regime. With the above rescaling, we shall argue that the lowest order of  $\rho_T$  does not exceed  $a^{-4}$  in the following discussion. We formulate the statement as a theorem.

**Theorem 1.** *In the high redshift regime ( $a \ll 1$ ),  $\rho_T = O(a^{-4})$ .*

*Proof.* First we expand

$$\tilde{H}^2(t) = \sum_{k=-c}^{\infty} r_k a^{k-4}, \quad (r_k < \infty) \quad (11)$$

where  $c$  is some integer, so that we have

$$\frac{d\tilde{H}}{dt} = \sum_{k=-c}^{\infty} \left( \frac{k-4}{2} \right) r_k a^{k-4}. \quad (12)$$

Using (7), (9), (11) and (12), we obtain

$$\tilde{R} = \frac{3\tilde{a}_1}{\tilde{\mu}} \left( \sum_{k=-c}^{\infty} k \cdot r_k a^{k-4} \right) + \frac{3\tilde{a}_0}{\tilde{\mu} a^3}, \quad (13)$$

$$\tilde{\Phi} = -\frac{3}{2} \tilde{H} \cdot \frac{\tilde{a}_1 \left( \sum_{k=-c}^{\infty} k(k-4) r_k a^{k-4} \right) - \frac{3\tilde{a}_0}{a^3}}{\tilde{a}_1 \left( \sum_{k=-c}^{\infty} k r_k a^{k-4} \right) + \frac{\tilde{a}_0}{a^3}}. \quad (14)$$

Substituting (11), (12), (13) and (14) into (8), and comparing the lowest power (requiring  $c > -1$ , otherwise losing its leading position) of  $a$  in the high redshift,  $a \ll 1$ , we derive the following relation

$$\left( \frac{\tilde{a}_1}{\tilde{\mu}} \right)^2 c^2 \cdot r_{-c}^3 \left[ c^2 + (5 - \frac{\tilde{a}_0}{\tilde{\mu}})c + 4 \right] = 0, \quad (15)$$

which leads to  $r_{-c} = 0$  if  $c \geq 1$  as  $0 < \tilde{a}_0/\tilde{\mu} < 1$  and  $c^2 + (5 - \frac{\tilde{a}_0}{\tilde{\mu}})c + 4 \neq 0$ . This is equivalent to say that (11) has the form

$$\tilde{H}^2 = \frac{r_0}{a^4} + \frac{r_1}{a^3} + \frac{r_2}{a^2} + \frac{r_3}{a} + r_4 + \dots \quad (16)$$

Finally, we achieve our claim from (2) that

$$\begin{aligned} \frac{\rho_T}{\rho_m^{(0)}} &= - \left( \frac{\chi}{a^4} + \frac{1}{a^4} \right) + \tilde{H}^2 \\ &= - \left( \frac{\chi}{a^4} + \frac{1}{a^3} \right) + \left( \frac{r_0}{a^4} + \frac{r_1}{a^3} + \frac{r_2}{a^2} + \frac{r_3}{a} + r_4 + \dots \right) = O \left( \frac{1}{a^4} \right). \end{aligned} \quad (17)$$

Note that the last equality follows since  $r_0 \neq \chi$ , which will be explained later. This is the end of the proof.  $\square$

We now write the expansion in (17), by the theorem above, simply as

$$\frac{\rho_T}{\rho_m^{(0)}} = \sum_{k=-4}^{\infty} A_{-k} a^k. \quad (18)$$

We shall only take first few dominating terms for a sufficient demonstration. By the procedure in the proof of the theorem, we can as well compare terms of various orders to yield the following relations,

$$O(a^{-10}) : 3(A_4 + \chi) \left(1 + \frac{\tilde{a}_1}{\tilde{\mu}} A_3\right)^2 = 0, \quad (19)$$

$$O(a^{-9}) : 2 \left(1 + \frac{\tilde{a}_1}{\tilde{\mu}} A_3\right) \left[ \frac{\tilde{a}_0}{\tilde{\mu}} \left(1 + \frac{\tilde{a}_1}{\tilde{\mu}} A_3\right) A_3 + \frac{4\tilde{a}_1}{\tilde{\mu}} (A_4 + \chi) A_2 \right] = 0, \quad (20)$$

$$O(a^{-8}) : \left(1 + \frac{\tilde{a}_1}{\tilde{\mu}} A_3\right) \left[ 4 \frac{\tilde{a}_0}{\tilde{\mu}} \left(1 + 3 \frac{\tilde{a}_1}{\tilde{\mu}} A_3\right) A_2 - \left(3A_2 + \frac{\tilde{a}_1}{\tilde{\mu}} (A_2 (2 + 5A_3) - 18A_1 (A_4 + \chi))\right) \right] = 0. \quad (21)$$

From (19), (20) and (21), one concludes a relation,

$$A_3 = -\frac{\tilde{\mu}}{\tilde{a}_1} = -\frac{(\tilde{a}_0 + \tilde{a}_1)}{\tilde{a}_1} < -1, \quad (22)$$

with  $A_1, A_2$  and  $A_4$  left as arbitrary constants to be determined by initial conditions and (10). Note that (22) implies  $r_1 = -\tilde{a}_0/\tilde{a}_1 < 0$  in (11). However, due to the observational data that  $a = 1$  at the current stage, the radiation density is much smaller than the dust density ( $\chi \ll 1$ ), whereas the torsion density is the same order as the dust density, as seen from (17),

$$\frac{\rho_T^{(0)}}{\rho_m^{(0)}} = [(r_0 - \chi) + r_2 + \dots] - (1 + |r_1|) \simeq O(1). \quad (23)$$

Subsequently, we have that  $[(r_0 - \chi) + r_2 + \dots] \leq \max\{O(1), O(|r_1|)\}$ , along with the assumption  $r_k < \infty$  for each  $k$ . As a result, we conclude that  $r_k$ , for all  $k \neq 1$ , should not be too large, which forbids the possibility  $r_0 = \chi$ . This argument shows the validity of the last equality in (17) with the non-vanishing  $O(1/a^4)$  coefficient.

From (4), via the continuity equation [13], we obtain

$$w_T = -1 - \frac{\rho'_T}{3\rho_T} \simeq -1 + \frac{1}{3} \left( \frac{4A_4 a^{-4} + 3A_3 a^{-3}}{A_4 a^{-4} + A_3 a^{-3}} \right) \simeq \frac{1}{3} \left( 1 - \frac{A_3}{A_4} a \right), \quad (24)$$

where the prime “ $\prime$ ” stands for  $d/d \ln a$  and we have used (18) for  $a \ll 1$ .

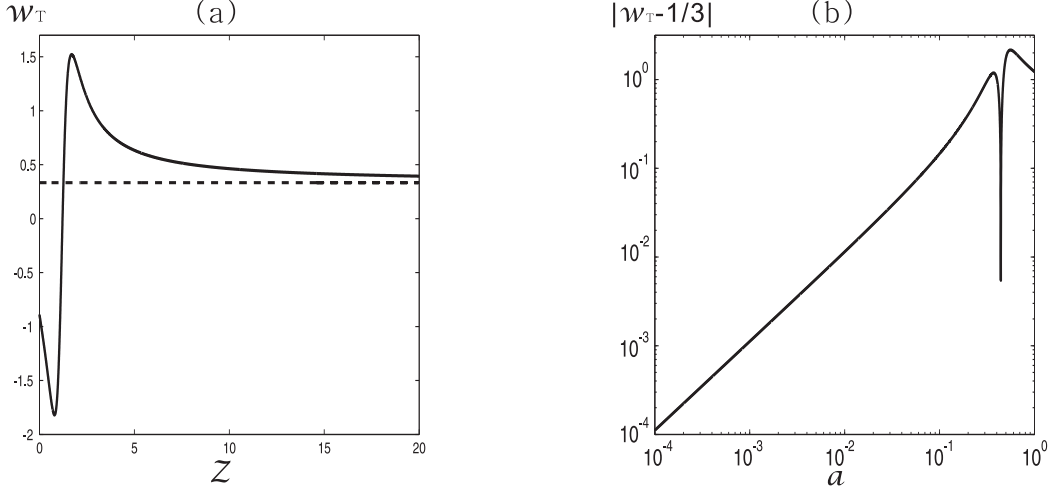


FIG. 1. Evolutions of (a)  $w_T$  and (b)  $|w_T - 1/3|$  as function of the redshift  $z$  and the scale parameter  $a$ , respectively, where the parameters and initial conditions are chosen as  $\tilde{a}_0 = 2$ ,  $\tilde{a}_1 = 1$ ,  $\tilde{H}_0 = 2$ ,  $\tilde{R}_0 = 14$  and  $\chi = \rho_r^{(0)}/\rho_m^{(0)} = 3.1 \times 10^{-4}$ .

### C. Numerical computations

In this subsection, we perform numerical computations to support the analysis above. As an illustration, we take the parameters  $\tilde{a}_0 = 2$  and  $\tilde{a}_1 = 1$  and initial conditions

$$\tilde{H}(z=0) = \tilde{H}_0 = 2, \quad \tilde{R}(z=0) = \tilde{R}_0 = 14,$$

and show the evolutions of  $w_T$ ,  $\tilde{\Phi}$ , and  $\tilde{R}$  in Figs. 1, 2a and 2b, respectively.

In Fig. 1a, we demonstrate the EoS of torsion as a function of the redshift  $z$ . As seen from the figure, in the high redshift regime  $w_T$  approaches  $1/3$ , which indeed shows an asymptotic behavior. Fig. 1b indicates that  $|w_T - 1/3|$  approximates a straight line in the scale factor  $a$  in the log-scaled coordinate since the slope in the log-scaled coordinates is nearly 1. The singularity in the interval  $[0.1, 1]$  corresponds to the crossing  $1/3$  of  $w_T$ . Thus, the numerical results concur with our semi-analytical approximation in (24). In Fig. 2, we observe that the behaviors of  $\tilde{R}$  and  $\tilde{\Phi} \propto 1/a^2$  in the high redshift regime are consistent with the results in (13) and (14), given by

$$\tilde{R} \simeq \frac{2\tilde{a}_1}{\tilde{\mu}} A_2 a^{-2}, \quad (25)$$

$$\tilde{\Phi} \simeq 3\tilde{H} \propto a^{-2}, \quad (26)$$

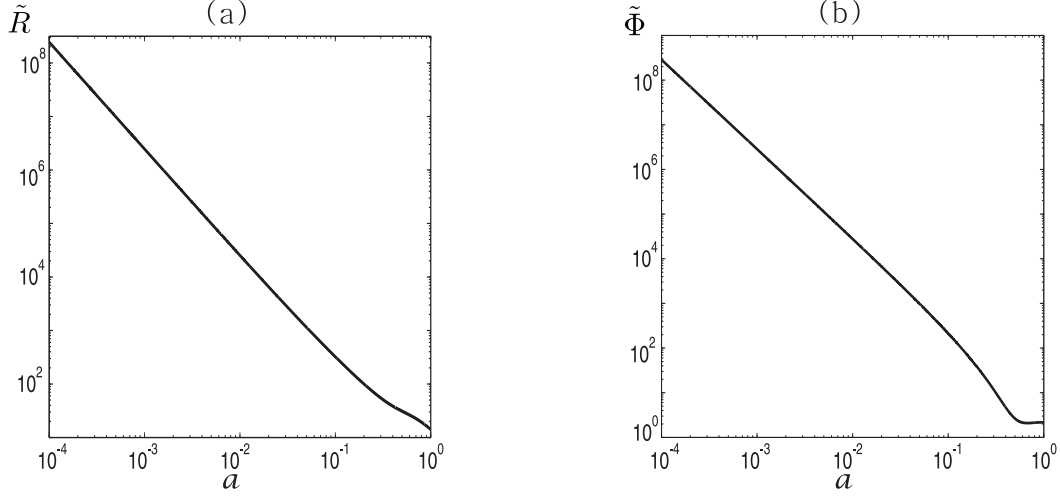


FIG. 2. Evolutions of (a) the rescaled affine curvature  $\tilde{R}$  and (b) the torsion  $\tilde{\Phi}$  as functions of the scale parameter  $a$  in the log scale with the parameters and initial conditions taken to be the same as Fig. 1.

respectively, where  $\tilde{H}^2 \simeq (\chi + A_4) a^{-4}$  from (11). Note that from (25), the behavior of the affine curvature  $\tilde{R}$  is highly different from that of the Riemannian scalar curvature  $\bar{R} = -\mathcal{T}/a_0 = \rho_m/a_0$ , which is proportional to  $1/a^3$  in both matter (dust) and radiation dominated eras.

### III. CONCLUSIONS

We have investigated the asymptotic evolution behaviors of the scalar-torsion mode in PGT. The EoS of the torsion density has an early time stable point  $w_T(z \gg 0) \rightarrow 1/3$ . This behavior can be estimated through the semi-analytical solution via the Laurent expansion in the scale factor  $a(t)$  for the torsion density  $\rho_T$ . We have shown that there indeed exists the lowest degree of  $\rho_T$  in its expansion by  $a^{-4}$ , corresponding to the radiation-like behavior in the high redshift regime. By the comparison of the next leading-order term of  $a^{-3}$  in the field equations, we have extracted the coefficient  $A_3 = -\mu/a_1$ , which results in the vanishing of the  $a^{-3}$  term in the affine curvature  $R$ , such that  $R$  is only proportional to  $a^{-2}$ , consistent with the numerical demonstration.

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